10-3 Operations With Radical Expressions

Common Core State Standards

Prepares for A-REI.A.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

MP 1, MP 2, MP 3, MP 4, MP 7

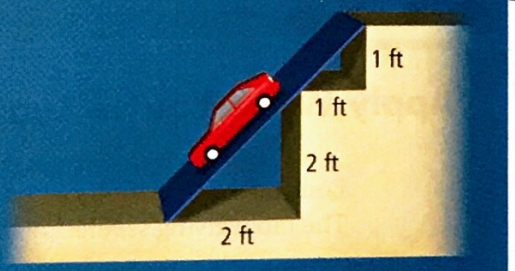
Objectives To simplify sums and differences of radical expressions
To simplify products and quotients of radical expressions

Notice that the ramp is the hypotenuse of two right triangles. This should help you get started.

SOLVE IT!

Getting Ready!

A volunteer is building a new ramp for a toy car derby. The ramp takes the car down the two steps shown in the diagram. How long should the volunteer make the ramp? Justify your reasoning.



MATHEMATICAL PRACTICES

Essential Understanding You can use properties of real numbers to perform operations with radical expressions.

For example, you can use the Distributive Property to simplify sums or differences of radical expressions by combining *like radicals*. **Like radicals,** such as $3\sqrt{5}$ and $7\sqrt{5}$, have the same radicand. **Unlike radicals,** such as $4\sqrt{3}$ and $-2\sqrt{2}$, have different radicands.

VOCABULARY

Lesson Vocabulary
- like radicals
- unlike radicals
- conjugates

Problem 1 Combining Like Radicals

What is the simplified form of each expression?

A $6\sqrt{11} + 9\sqrt{11}$

$6\sqrt{11} + 9\sqrt{11} = (6 + 9)\sqrt{11}$ Use the Distributive Property to combine like radicals.

$= 15\sqrt{11}$ Simplify.

B $\sqrt{3} - 5\sqrt{3}$

$\sqrt{3} - 5\sqrt{3} = 1\sqrt{3} - 5\sqrt{3}$ Write $\sqrt{3}$ as $1\sqrt{3}$.

$= (1 - 5)\sqrt{3}$ Use the Distributive Property to combine like radicals.

$= -4\sqrt{3}$ Simplify.

Think

Have you seen a problem like this before?
Yes. Combining like radicals is similar to combining like terms. For example, simplifying the expression in part (A) is similar to simplifying $6x + 9x$.

Got It? **1.** What is the simplified form of each expression?

a. $7\sqrt{2} - 8\sqrt{2}$ **b.** $5\sqrt{5} + 2\sqrt{5}$

You may need to simplify radical expressions first to determine if they can be added or subtracted by combining like radicals.

Problem 2 Simplifying to Combine Like Radicals

What is the simplified form of $5\sqrt{3} - \sqrt{12}$?

Think
How do you know if radical expressions can be combined?
Simplify all radicals. Although $\sqrt{3}$ and $\sqrt{12}$ are unlike radicals, they can be combined after $\sqrt{12}$ is simplified.

$5\sqrt{3} - \sqrt{12} = 5\sqrt{3} - \sqrt{4 \cdot 3}$ 4 is a perfect-square factor of 12.

$= 5\sqrt{3} - \sqrt{4} \cdot \sqrt{3}$ Multiplication Property of Square Roots

$= 5\sqrt{3} - 2\sqrt{3}$ Simplify $\sqrt{4}$.

$= (5 - 2)\sqrt{3}$ Use the Distributive Property to combine like radicals.

$= 3\sqrt{3}$ Simplify.

Got It? **2.** What is the simplified form of each expression in parts (a) and (b)?

a. $4\sqrt{7} + 2\sqrt{28}$ **b.** $5\sqrt{32} - 4\sqrt{18}$

c. Reasoning Can you combine two unlike radicals when the radicands have no common factors other than 1? Explain.

When simplifying a product like $\sqrt{10}(\sqrt{6} + 3)$, you can use the Distributive Property to multiply $\sqrt{10}$ times $\sqrt{6}$ and $\sqrt{10}$ times 3. If both factors in the product have two terms, as in $(\sqrt{6} - 2\sqrt{3})(\sqrt{6} + \sqrt{3})$, you can use FOIL to multiply just as you do when multiplying binomials.

Problem 3 Multiplying Radical Expressions

What is the simplified form of each expression?

Think
Have you seen a problem like this before?
Yes. Parts (A) and (B) are similar to simplifying products like $3(x + 2)$ and $(2x + 1)(x - 5)$.

A $\sqrt{10}(\sqrt{6} + 3)$

$\sqrt{10}(\sqrt{6} + 3) = (\sqrt{10} \cdot \sqrt{6}) + (\sqrt{10} \cdot 3)$ Distributive Property

$= \sqrt{60} + 3\sqrt{10}$ Multiplication Property of Square Roots

$= \sqrt{4} \cdot \sqrt{15} + 3\sqrt{10}$ 4 is a perfect-square factor of 60.

$= 2\sqrt{15} + 3\sqrt{10}$ Simplify $\sqrt{4}$.

B $(\sqrt{6} - 2\sqrt{3})(\sqrt{6} + \sqrt{3})$

$(\sqrt{6} - 2\sqrt{3})(\sqrt{6} + \sqrt{3}) = \sqrt{36} + \sqrt{18} - 2\sqrt{18} - 2\sqrt{9}$ Use FOIL.

$= 6 - \sqrt{18} - 2(3)$ Combine like radicals and simplify.

$= 6 - \sqrt{9} \cdot \sqrt{2} - 6$ 9 is a perfect-square factor of 18.

$= -3\sqrt{2}$ Simplify.

Got It? **3.** What is the simplified form of each expression?

a. $\sqrt{2}(\sqrt{6} + 5)$ **b.** $(\sqrt{11} - 2)^2$ **c.** $(\sqrt{6} - 2\sqrt{3})(4\sqrt{3} + 3\sqrt{6})$

Conjugates are the sum and difference of the same two terms. For example, $\sqrt{7} + \sqrt{3}$ and $\sqrt{7} - \sqrt{3}$ are conjugates. The product of conjugates is a difference of squares.

$$(\sqrt{7} + \sqrt{3})(\sqrt{7} - \sqrt{3}) = (\sqrt{7})^2 - (\sqrt{3})^2$$
$$= 7 - 3 = 4$$

The product of the conjugates has no radicals.

You can use conjugates to simplify a quotient whose denominator is a sum or difference of radicals.

Problem 4 Rationalizing a Denominator Using Conjugates

What is the simplified form of $\frac{10}{\sqrt{7} - \sqrt{2}}$?

Plan

How do you rationalize the denominator?
Multiply by the conjugate of the denominator. If the denominator has the form $a - b$, the conjugate is $a + b$.

$$\frac{10}{\sqrt{7} - \sqrt{2}} = \frac{10}{\sqrt{7} - \sqrt{2}} \cdot \frac{\sqrt{7} + \sqrt{2}}{\sqrt{7} + \sqrt{2}}$$ Multiply the numerator and denominator by the conjugate of the denominator.

$$= \frac{10(\sqrt{7} + \sqrt{2})}{7 - 2}$$ Multiply in the denominator.

$$= \frac{10(\sqrt{7} + \sqrt{2})}{5}$$ Simplify the denominator.

$$= 2(\sqrt{7} + \sqrt{2})$$ Divide 10 and 5 by the common factor 5.

$$= 2\sqrt{7} + 2\sqrt{2}$$ Simplify the expression.

Got It? 4. What is the simplified form of $\frac{-3}{\sqrt{10} + \sqrt{5}}$?

Golden rectangles appear frequently in nature and art. The ratio of the length to the width of a golden rectangle is $(1 + \sqrt{5}) : 2$.

Problem 5 Solving a Proportion Involving Radicals STEM

Biology **Fiddlehead ferns naturally grow in spirals that fit into golden rectangles. What is the width w of the fern shown?**

Think

How do you begin this problem?
Since the rectangle is a golden rectangle, the length divided by the width has to equal $\frac{1 + \sqrt{5}}{2}$.

$$\frac{1 + \sqrt{5}}{2} = \frac{4}{w}$$ Write a proportion.

$$w(1 + \sqrt{5}) = 8$$ Cross Products Property

$$w = \frac{8}{1 + \sqrt{5}}$$ Divide each side by $1 + \sqrt{5}$.

$$w = \frac{8}{1 + \sqrt{5}} \cdot \frac{1 - \sqrt{5}}{1 - \sqrt{5}}$$ Multiply the numerator and denominator by the conjugate of the denominator.

$$w = \frac{8 - 8\sqrt{5}}{1 - 5}$$ Multiply.

$$w = \frac{8 - 8\sqrt{5}}{-4}$$ Simplify the denominator.

$$w = -2 + 2\sqrt{5} \approx 2.5$$ Simplify. Use a calculator.

The width of the fern is about 2.5 cm.

Got It? 5. A golden rectangle is 12 in. long. What is the width of the rectangle? Write your answer in simplified radical form. Round to the nearest tenth of an inch.

Lesson Check

Do you know HOW?

Simplify each radical expression.

1. $4\sqrt{3} + \sqrt{3}$
2. $3\sqrt{6} - \sqrt{24}$
3. $\sqrt{7}(\sqrt{3} - 2)$
4. $(\sqrt{5} - 6)^2$
5. $\frac{7\sqrt{5}}{3 + \sqrt{2}}$
6. $\frac{6}{\sqrt{7} + 2}$

Do you UNDERSTAND? MATHEMATICAL PRACTICES

7. **Vocabulary** What is the conjugate of each expression?
   a. $\sqrt{13} - 2$ b. $\sqrt{6} + \sqrt{3}$ c. $\sqrt{5} - \sqrt{10}$
8. **Error Analysis** A student simplified an expression, as shown below. Describe and correct the error.

$$\frac{1}{\sqrt{3}-1} = \frac{1}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\sqrt{3}+1}{9-1} = \frac{\sqrt{3}+1}{8}$$

Practice and Problem-Solving Exercises MATHEMATICAL PRACTICES

A Practice

Simplify each sum or difference. See Problems 1 and 2.

9. $\sqrt{5} + 6\sqrt{5}$
10. $12\sqrt{5} - 3\sqrt{5}$
11. $7\sqrt{3} + \sqrt{3}$
12. $4\sqrt{2} - 7\sqrt{2}$
13. $3\sqrt{7} - \sqrt{63}$
14. $4\sqrt{128} + 5\sqrt{18}$
15. $3\sqrt{45} - 8\sqrt{20}$
16. $\sqrt{28} - 5\sqrt{7}$
17. $-6\sqrt{10} + 5\sqrt{90}$
18. $3\sqrt{3} - 2\sqrt{12}$
19. $-\frac{1}{2}\sqrt{5} + 2\sqrt{125}$
20. $5\sqrt{8} + 2\sqrt{72}$

Simplify each product. See Problem 3.

21. $\sqrt{6}(\sqrt{2} + \sqrt{3})$
22. $\sqrt{5}(\sqrt{15} - 3)$
23. $3\sqrt{7}(1 - \sqrt{7})$
24. $-\sqrt{12}(4 - 2\sqrt{3})$
25. $5\sqrt{11}(\sqrt{3} - 3\sqrt{2})$
26. $(3\sqrt{11} + \sqrt{7})^2$
27. $(2 + \sqrt{10})(2 - \sqrt{10})$
28. $(\sqrt{6} + \sqrt{3})(\sqrt{2} - 2)$
29. $(5\sqrt{2} - 2\sqrt{3})^2$

Simplify each quotient. See Problem 4.

30. $\frac{5}{\sqrt{2} - 1}$
31. $\frac{3}{\sqrt{7} - \sqrt{3}}$
32. $\frac{-2}{\sqrt{6} + \sqrt{11}}$
33. $\frac{\sqrt{5}}{2 - \sqrt{5}}$
34. $\frac{-1}{2 - 2\sqrt{3}}$
35. $\frac{7}{\sqrt{5} + \sqrt{13}}$

STEM 36. **Biology** A shell fits into a golden rectangle with a length of 8 in. What is the shell's width? Write your answer in simplified radical form and rounded to the nearest tenth of an inch. See Problem 5.

STEM 37. **Architecture** A room is approximately shaped like a golden rectangle. Its length is 23 ft. What is the room's width? Write your answer in simplified radical form and rounded to the nearest tenth of a foot.

B Apply

Find the exact solution for each equation. Find the approximate solution to the nearest tenth.

38. $\frac{5\sqrt{2}}{\sqrt{2}-1} = \frac{x}{\sqrt{2}}$

39. $\frac{3}{1+\sqrt{5}} = \frac{1-\sqrt{5}}{x}$

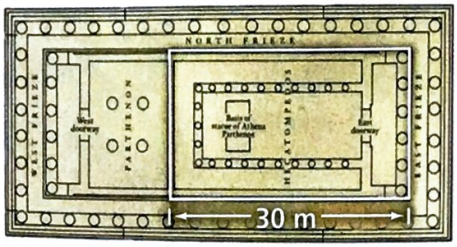
40. $\frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{x}{2}$

41. $\frac{x}{2+\sqrt{7}} = \frac{3-\sqrt{7}}{4}$

42. $\frac{4\sqrt{15}}{1+\sqrt{3}} = \frac{1+\sqrt{3}}{x}$

43. $\frac{2+\sqrt{2}}{2-\sqrt{2}} = \frac{x}{3+\sqrt{10}}$

44. History The floor plan of the Parthenon in Athens, Greece, is shown below. The marked room approximates a golden rectangle. What is the width of the room? Write your answer in simplified radical form. Round to the nearest tenth of a meter.



45. Writing Are $\sqrt{3}$ and $\sqrt{12}$ like radicals? Can their sum be simplified? Explain.

46. Error Analysis A student added two radical expressions as shown at the right. Describe and correct the student's mistake.

$\sqrt{6} + \sqrt{24} = \sqrt{6} + 4\sqrt{6}$
$= 5\sqrt{6}$

Simplify each expression.

47. $\sqrt{40} + \sqrt{90}$

48. $3\sqrt{2}(2+\sqrt{6})$

49. $\sqrt{12} + 4\sqrt{75} - \sqrt{36}$

50. $(\sqrt{3}+\sqrt{5})^2$

51. $\frac{\sqrt{13}+\sqrt{10}}{\sqrt{13}-\sqrt{5}}$

52. $(\sqrt{7}+\sqrt{8})(\sqrt{7}+\sqrt{8})$

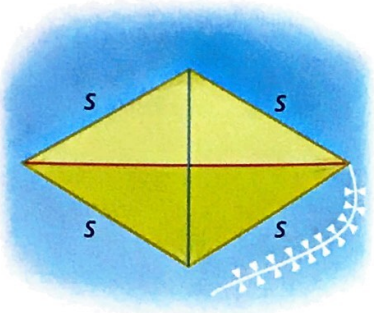
53. $2\sqrt{2}(-2\sqrt{32}+\sqrt{8})$

54. $4\sqrt{50} - 7\sqrt{18}$

55. $\frac{2\sqrt{12}+3\sqrt{6}}{\sqrt{9}-\sqrt{6}}$

STEM 56. Chemistry The ratio of the diffusion rates of two gases is given by the formula $\frac{r_1}{r_2} = \frac{\sqrt{m_2}}{\sqrt{m_1}}$, where m_1 and m_2 are the masses of the molecules of the gases. Find $\frac{r_1}{r_2}$ if $m_1 = 12$ units and $m_2 = 30$ units. Write your answer in simplified radical form.

57. Reasoning The diagram at the right shows the dimensions of a kite. The length of the vertical blue crosspiece is s. What is the length of the horizontal red crosspiece in terms of s?



58. Think About a Plan The formula $r = \sqrt{\frac{A}{P}} - 1$ gives the interest rate r, expressed as a decimal, that will allow principal P to grow into amount A in 2 yr, if the interest is compounded annually. If you invest \$10,000 and want to make \$2000 in interest over 2 yr, what interest rate do you need?

- What amount do you want in the account after 2 yr?
- What radical expression gives the interest rate you need?

59. a. Suppose n is an even number. Simplify $\sqrt{x^n}$.
b. Suppose n is an odd number greater than 1. Simplify $\sqrt{x^n}$.

60. Reasoning Simplify $\frac{a\sqrt{b}}{b\sqrt{a}}$.

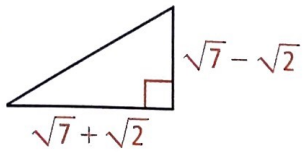
Challenge

61. Geometry A square has sides with length n. How much must be added to the length of one side to transform the square into a golden rectangle?

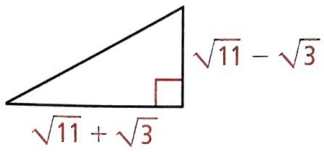
62. Reasoning What are three fractions that you can multiply $\frac{1}{\sqrt{2}+3}$ by to rationalize the denominator? Will the resulting products be the same? Explain.

63. Geometry Find the length of each hypotenuse. Write your answer in simplified radical form.

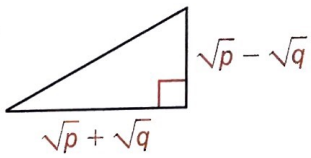
a.



b.



c.



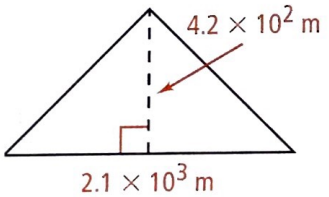
Standardized Test Prep

SAT/ACT

64. What is the simplified form of $2\sqrt{18}-\sqrt{32}+4\sqrt{8}$?

(A) $8\sqrt{3}$ (B) $10\sqrt{2}$ (C) $18\sqrt{2}$ (D) $10\sqrt{18}$

65. A surveyor is calculating the areas of lots that are going to be sold. The dimensions of one lot are shown at the right. What is the area of the lot shown?



(F) $8.82 \times 10^6\ m^2$ (H) $4.41 \times 10^5\ m^2$

(G) $8.82 \times 10^5\ m^2$ (I) $4.41 \times 10^6\ m^2$

66. What are the approximate solutions of the equation $\frac{5}{2}x^2+\frac{3}{4}x-5=0$? Use a graphing calculator.

(A) $-5, 0$ (B) $-1.57, 1.27$ (C) $-1.36, 0.71$ (D) $-0.96, 0.84$

Short Response

67. What are the domain and range of the function $y=|x|$? Show how you find your answer.

Mixed Review

Simplify each radical expression. See Lesson 10-2.

68. $\sqrt{108}$ **69.** $3\sqrt{150}$ **70.** $\frac{4}{\sqrt{18c^2}}$ **71.** $\sqrt{5}\cdot\sqrt{45}$

Rewrite each expression using each base only once. See Lesson 7-2.

72. $8^5\cdot 8^{11}$ **73.** $2^{24}\cdot 2^{-13}$ **74.** $5^{11}\cdot 5^{16}$ **75.** $3^7\cdot 3^{-4}$

Get Ready! To prepare for Lesson 10-4, do Exercises 76–81.

Solve by factoring. See Lesson 9-4.

76. $x^2+2x+1=0$ **77.** $x^2+x-12=0$ **78.** $x^2+2x-15=0$

79. $3x^2+7x-6=0$ **80.** $2x^2+3x-2=0$ **81.** $x^2+14x+49=0$