7-5 Rational Exponents and Radicals

Common Core State Standards

N-RN.A.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

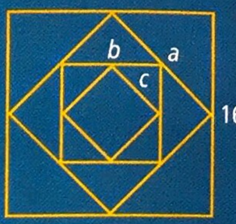
MP 1, MP 3, MP 4, MP 6, MP 7

Objective To rewrite expressions involving radicals and rational exponents

Make a sketch and look for right triangles. Look for a pattern in your answers and predict the value of c before finding the value of c.

SOLVE IT! **Getting Ready!**

The figure at the right is made up of squares. The side of a larger square is bisected by the vertices of the square that is one size smaller. What are the values of a, b, and c? (Hint: The hypotenuse of an isosceles right triangle equals a smaller side times $\sqrt{2}$.)

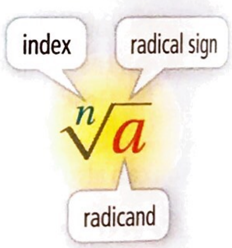


MATHEMATICAL PRACTICES In this lesson, you will learn the relationship between radical expressions and expressions using rational exponents.

Lesson Vocabulary
- index

Essential Understanding You can use rational exponents to represent radicals.

In a radical expression, the number under the radical sign a is the *radicand*. The number n in the crook of the radical sign is the *index*. The **index** gives the degree of the root. For a cube root, the degree is 3. If there is no index, the degree is 2, which means square root.



Recall what you know about square roots. Since $5^2 = 25$, you know that $\sqrt{25} = 5$. You also know that $25^{\frac{1}{2}} = 5$. Using the transitive property of equality, you can conclude that $\sqrt{25} = 25^{\frac{1}{2}}$. Similarly, $\sqrt[3]{8} = 8^{\frac{1}{3}}$.

You can simplify radical expressions by finding like factors, just as when simplifying powers with rational exponents.

Problem 1 Finding Roots

Think

How do you find $\sqrt[n]{a}$ for any number a?
You look for n equal factors of a.

What is the simplified form of each expression?

A $\sqrt[3]{125}$

Method 1

$$\sqrt[3]{125} = \sqrt[3]{5 \cdot 5 \cdot 5}$$
$$= 5$$

B $\sqrt[4]{16}$

Method 2

$$\sqrt[4]{16} = 16^{\frac{1}{4}}$$
$$= (2 \cdot 2 \cdot 2 \cdot 2)^{\frac{1}{4}}$$
$$= 2$$

Got It? 1. What is the simplified form of each expression?

a. $\sqrt[3]{27}$ b. $\sqrt[5]{32}$ c. $\sqrt[3]{64}$ d. $\sqrt[2]{36}$

You can also write expressions that have rational exponents like $\frac{2}{3}$ in radical form.

$8^{\frac{2}{3}} = 8^{2 \cdot \frac{1}{3}} = (8^2)^{\frac{1}{3}} = \sqrt[3]{8^2}$

$8^{\frac{2}{3}} = 8^{\frac{1}{3} \cdot 2} = (8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2$

So, $8^{\frac{2}{3}} = \sqrt[3]{8^2} = (\sqrt[3]{8})^2$.

take note

Key Concept Equivalence of Radicals and Rational Exponents

If the *n*th root of *a* is a real number and *m* and *n* are positive integers, then
$a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.

Problem 2 Converting to Radical Form

Plan

In radical form, do you use the numerator or denominator as the index?
You use the denominator of the rational exponent as the index of the radical.

A What is $12a^{\frac{2}{3}}$ in radical form?

$12a^{\frac{2}{3}} = 12\sqrt[3]{a^2}$ Rewrite $a^{\frac{2}{3}}$ in radical form.

B What is $(64a)^{\frac{4}{5}}$ in radical form?

$(64a)^{\frac{4}{5}} = (32 \cdot 2a)^{\frac{4}{5}}$ Since 32 is the 5th power of 2, write 64*a* as a product of 32 and 2*a*.

$= 32^{\frac{4}{5}}(2a)^{\frac{4}{5}}$ Power of a product

$= 2^{5 \cdot \frac{4}{5}} \cdot (2a)^{\frac{4}{5}}$ Rewrite 32 as 2^5.

$= 2^4 \cdot (2a)^{\frac{4}{5}}$ Simplify $5 \cdot \frac{4}{5}$.

$= 16\sqrt[5]{(2a)^4}$ Simplify and write $(2a)^{\frac{4}{5}}$ in radical form.

Got It? 2. What is each exponential expression in radical form?

a. $a^{\frac{5}{6}}$ b. $5x^{\frac{1}{3}}$ c. $(54y)^{\frac{2}{3}}$

Problem 3 Converting to Exponential Form

Think

Can you write $\sqrt[3]{27d^5}$ as $27d^{\frac{5}{3}}$?
No. Since the coefficient 27 is under the radical, the index operates on the coefficient as well as the variable.

A What is $\sqrt[5]{b^3}$ in exponential form?

$\sqrt[5]{b^3} = b^{\frac{3}{5}}$ Rewrite using exponential form.

B What is $\sqrt[3]{27d^5}$ in exponential form? Simplify.

$\sqrt[3]{27d^5} = (27d^5)^{\frac{1}{3}}$ Rewrite the radical expression in exponential form.

$= 27^{\frac{1}{3}}(d^5)^{\frac{1}{3}}$ Power of a product

$= 3d^{\frac{5}{3}}$ Simplify.

Got It? **3.** Write each radical expression in exponential form.

a. $\sqrt[3]{s^2}$ **b.** $12\sqrt[3]{x^4}$ **c.** $\sqrt{(4y)^5}$ **d.** $\sqrt[4]{256a^8}$

Problem 4 Using a Radical Expression STEM

Biology You can estimate the metabolic rate of living organisms based on body mass using Kleiber's law. The formula $R = 73.3\sqrt[4]{M^3}$ relates metabolic rate R measured in Calories per day to body mass M measured in kilograms. What is the metabolic rate of a dog with a body mass of 18 kg?

Think

How can you find the approximate value of the expression?
You can use 18^(3/4) to simplify the radical using a calculator.

$R = 73.3\sqrt[4]{M^3}$

$= 73.3\sqrt[4]{18^3}$ Substitute 18 for M.

≈ 640.5578436 Use a calculator to simplify.

The metabolic rate is about 641 Calories per day.

Got It? **4.** What is the metabolic rate of a man with a body mass of 75 kg?

Lesson Check

Do you know HOW?

Simplify each expression.

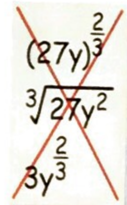
1. $\sqrt[6]{64}$ **2.** $\sqrt[4]{81}$ **3.** $\left(\sqrt[3]{125}\right)^4$

Write each expression using rational exponents in radical form and each radical expression in exponential form.

4. $\sqrt{x}$ **5.** $c^{\frac{1}{5}}$ **6.** $(8d)^{\frac{2}{3}}$ **7.** $\sqrt[4]{16y^3}$

Do you UNDERSTAND? MATHEMATICAL PRACTICES

8. Error Analysis What is the error in the problem at the right? What is the correct answer?



9. Write a rule for multiplying two radicals with the same radicand. Justify why your rule works.

10. Does $\sqrt{4^3} - \sqrt{4} = 4$? Explain why or why not.

Practice and Problem-Solving Exercises MATHEMATICAL PRACTICES

A Practice

What is the value of each expression? See Problem 1.

11. $\sqrt[2]{49}$ **12.** $\sqrt[5]{1}$ **13.** $\sqrt[4]{625}$

14. $\sqrt[2]{81}$ **15.** $\sqrt[3]{216}$ **16.** $\sqrt[4]{81}$

Write each expression in radical form. See Problem 2.

17. $a^{\frac{2}{3}}$ **18.** $(64b)^{\frac{3}{4}}$ **19.** $25x^{\frac{1}{2}}$

20. $z^{\frac{3}{4}}$ **21.** $(25x)^{\frac{1}{2}}$ **22.** $27a^{\frac{2}{3}}$

23. $(98d)^{\frac{1}{2}}$ **24.** $18b^{\frac{1}{4}}$ **25.** $(24c)^{\frac{2}{3}}$

Write each expression in exponential form.

See Problem 3.

26. $\sqrt[5]{a^3}$ 27. $\sqrt{(2c)^4}$ 28. $\sqrt[4]{256a^3}$

29. $\sqrt[3]{(8x)^2}$ 30. $\sqrt[3]{27c^2}$ 31. $\sqrt[4]{625y^3}$

32. $\sqrt{36x}$ 33. $\sqrt[4]{x^3}$ 34. $\sqrt[3]{8b^5}$

35. **Manufacturing** A company that manufactures memory chips for digital cameras uses the formula $c = 120\sqrt[3]{n^2} + 1300$ to determine the cost c, in dollars, of producing n chips. How much will it cost to produce 250 chips?

See Problem 4.

STEM 36. **Archaeology** Carbon-14 is present in all living organisms and decays at a predictable rate. To estimate the age of an organism, archaeologists measure the amount of carbon-14 left in its remains. The approximate amount of carbon-14 remaining after 5000 years can be found using the formula $A = A_0(2.7)^{-\frac{3}{5}}$, where A_0 is the initial amount of carbon-14 in the sample that is tested. How much carbon-14 is left in a sample that is 5000 years old and originally contained 7.0×10^{-12} grams of carbon-14?

B Apply

Simplify each expression using the properties of exponents, and then write the expression in radical form.

37. $\left(x^{\frac{3}{4}}\right)\left(x^{\frac{1}{2}}\right)$ 38. $\left(a^{\frac{2}{3}}\right)\left(a^{\frac{1}{4}}\right)$ 39. $(cd)^{\frac{1}{2}}\left(d^{\frac{1}{3}}\right)$

40. $(3x^{\frac{1}{3}})(8x^2)$ 41. $(36x)^{\frac{1}{2}}(49x)^{\frac{1}{2}}$ 42. $\left(x^{\frac{2}{3}}\right)(8x)^{\frac{1}{3}}$

Write each expression in exponential form. Simplify when possible.

43. $\sqrt[3]{b^2} - \sqrt[3]{b}$ 44. $3\sqrt[4]{a^3} - 2\sqrt[4]{a^3}$ 45. $\left(\sqrt[3]{8b^5}\right) - \left(\sqrt[4]{256a^3}\right)$

46. $\sqrt[4]{(9x)^2} + \sqrt[4]{625y^3}$ 47. $\left(\sqrt[3]{y}\right)\left(\sqrt[3]{y}\right)\left(\sqrt[3]{y}\right)$ 48. $\sqrt{(2c)^4} + \sqrt[3]{c^6}$

49. **Sports** The radius r of a sphere that has volume V is $r = \sqrt[3]{\frac{3V}{4\pi}}$. The volume of a basketball is approximately 434.67 in.^{3}. The radius of a tennis ball is about one fourth the radius of a basketball. Find the radius of the tennis ball.

50. a. Show that $\sqrt{x^2} = x$ by rewriting $\sqrt{x^2}$ in exponential form.

    b. Show that $\sqrt[4]{x^2} = \sqrt{x}$ by rewriting $\sqrt[4]{x^2}$ in exponential form.

51. **Think about a Plan** You want to simplify the expression $4x^{\frac{3}{2}} + 3\sqrt{x^3}$.
    - How can you write the radical expression using a rational exponent?
    - Can you add the resulting terms?
    - What is the result in simplest form?
    - Can you write the result in two equivalent forms?

52. **Open-Ended** Write an expression using rational exponents. Then write an equivalent expression using radicals.

53. **Inflation** The formula $C = c(1 + r)^n$ can be used to estimate the future cost C of an item due to inflation. Here c represents the current cost of the item, r is the rate of inflation, and n is the number of years for the projection. Suppose a video game system costs \$299 now. How much will the price increase in nine months with an annual inflation rate of 3.2%?

Challenge

54. **Cells** The number of cells in a cell culture grows exponentially. The number of cells in the culture as a function of time is given by the expression $N\left(\frac{6}{5}\right)^t$, where t is measured in hours and N is the initial size of the culture.
   a. After 2 hours, there were 144 cells in the culture. What was N?
   b. How many cells were in the culture after 20 minutes?
   c. How many cells were in the culture after 2.5 hours?

Standardized Test Prep

55. Which of the following expressions is equivalent to $(8x)^{\frac{4}{3}}$?

   (A) $16\sqrt[3]{x^4}$ (C) $\sqrt[3]{8x^4}$
   (B) $\sqrt[4]{16x^3}$ (D) $8\sqrt[4]{x^3}$

56. Which of the following expressions is equivalent to $4\sqrt{b^5}$?

   (F) $2b^{\frac{2}{5}}$ (H) $(2b)^{\frac{2}{5}}$
   (G) $(4b)^{\frac{5}{2}}$ (I) $4b^{\frac{5}{2}}$

Short Response

57. Write the expression $\sqrt{9s^3} + \sqrt{16s^3}$ with rational exponents.

Mixed Review

Simplify the following expressions. See Lesson 7-3.

58. $5(t^2)^3$ 59. $(-2x^4)^5$ 60. $-6(a^0)^9$ 61. $\left((9d)^{\frac{1}{2}}\right)^3$

62. $-12(c^2)^1$ 63. $(10a^3)^2$ 64. $-4(y^4)^2$ 65. $\left((27t)^{\frac{1}{3}}\right)^2$

Write each equation in standard form. See Lesson 5-5.

66. $y = 4x + 7$ 67. $y + 3 = 2(x - 5)$ 68. $-3y + 6 = \frac{7x}{2}$

69. $3y - 5 = 6(x + 2)$ 70. $x = 6y + 3$ 71. $9y = \frac{9x}{4} + 27$

Get Ready! **To Prepare for Lesson 7-6, do Exercises 72–77.**

Describe the pattern for each sequence. Then write the next three numbers in the sequence. See Lesson 4-7.

72. 3, 6, 9, . . . 73. 10, 4, −2, . . . 74. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

75. 2, 7, 12, . . . 76. 3, −1, −5, . . . 77. 1, 4, 9, . . .