7-7 Exponential Growth and Decay

Common Core State Standards

F-IF.C.8b Use the properties of exponents to interpret expressions for exponential functions . . . **Also A-SSE.B.3c, A-CED.A.2, F-LE.A.1c, F-LE.B.5**

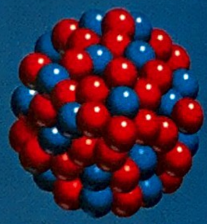
MP 1, MP 2, MP 3, MP 4, MP 8

Objective To model exponential growth and decay

SOLVE IT! Getting Ready!

The half-life of a radioactive substance is the length of time it takes for half of the atoms in a sample of the substance to decay. The half-life of uranium-238 is 4.46×10^9 yr.

Suppose you have a sample of 1000 uranium-238 atoms. How many atoms of uranium-238 are left after 1.338×10^{10} yr? Explain your reasoning.



Uranium-238

MATHEMATICAL PRACTICES In the Solve It, the number of uranium-238 atoms decreases exponentially. In this lesson, you will use exponential functions to model similar situations.

Essential Understanding An exponential function can model growth or decay of an initial amount.

Lesson Vocabulary
- exponential growth
- growth factor
- compound interest
- exponential decay
- decay factor

take note

Key Concept Exponential Growth

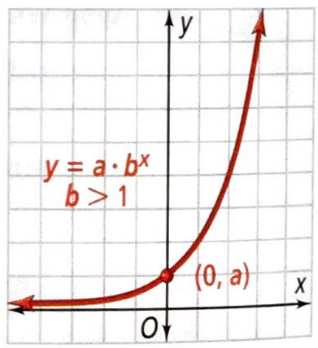
Definitions

Exponential growth can be modeled by the function $y = a \cdot b^x$, where $a > 0$ and $b > 1$. The base b is the **growth factor,** which equals 1 plus the percent rate of change expressed as a decimal.

Algebra

initial amount (when $x = 0$)
↓
$y = a \cdot b^x$ ← exponent
↑
The base, which is greater than 1, is the growth factor.

Graph



Problem 1 Modeling Exponential Growth

Economics Since 2005, the amount of money spent at restaurants in the United States has increased about 7% each year. In 2005, about \$360 billion was spent at restaurants.

Think

When can you use an exponential growth function?

You can use an exponential growth function when an initial amount increases by a fixed percent each time period.

A **If the trend continues, about how much will be spent at restaurants in 2015?**

Relate $y = a \cdot b^x$ Use an exponential function.

Define Let x = the number of years since 2005.
Let y = the annual amount spent at restaurants (in billions of dollars).
Let a = the initial amount spent (in billions of dollars), 360.
Let b = the growth factor, which is $1 + 0.07 = 1.07$.

Write $y = 360 \cdot 1.07^x$

Use the equation to predict the annual spending in 2015.

$y = 360 \cdot 1.07^x$

$= 360 \cdot 1.07^{10}$ 2015 is 10 yr after 2005, so substitute 10 for x.

≈ 708 Round to the nearest billion dollars.

About \$708 billion will be spent at restaurants in the United States in 2015 if the trend continues.

B **What is an expression that represents the equivalent monthly increase of spending at U.S. restaurants in 2005?**

You will need to find an expression of the form r^m, where r is approximately the monthly growth factor and m is the number of months. You know that 1.07^x represents the yearly increase where x is the number of years.

$1.07^x = 1.07^{\frac{12x}{12}}$ There are $12x$ months in x years.

$= \left(1.07^{\frac{1}{12}}\right)^{12x}$ Power raised to a power

$\approx 1.0057^{12x}$ Simplify.

$= 1.0057^m$ Let $12x = m$, the number of months.

The expression 1.0057^m represents the equivalent monthly increase of spending at restaurants.

Got It? **1.** Suppose that in 1985, there were 285 cell phone subscribers in a small town. The number of subscribers increased by 75% each year after 1985. How many cell phone subscribers were in the small town in 1994? Write an expression to represent the equivalent monthly cell phone subscription increase.

When a bank pays interest on both the principal *and* the interest an account has already earned, the bank is paying **compound interest.** Compound interest is an example of exponential growth.

You can use the following formula to find the balance of an account that earns compound interest.

$A = P\left(1 + \frac{r}{n}\right)^{nt}$

A = the balance
P = the principal (the initial deposit)
r = the annual interest rate (expressed as a decimal)
n = the number of times interest is compounded per year
t = the time in years

Problem 2 Compound Interest

Finance **Suppose that when your friend was born, your friend's parents deposited \$2000 in an account paying 4.5% interest compounded quarterly. What will the account balance be after 18 yr?**

Know
- \$2000 principal
- 4.5% interest
- interest compounded quarterly

Need
Account balance in 18 yr

Plan
Use the compound interest formula.

Think
Is the formula an exponential growth function?
Yes. You can rewrite the formula as $A = P\left[\left(1 + \frac{r}{n}\right)^n\right]^t$. So it is an exponential function with initial amount P and growth factor $\left(1 + \frac{r}{n}\right)^n$.

$A = P(1 + \frac{r}{n})^{nt}$	Use the compound interest formula.
$= 2000\left(1 + \frac{0.045}{4}\right)^{4 \cdot 18}$	Substitute the values for P, r, n, and t.
$= 2000(1.01125)^{72}$	Simplify.

The balance will be \$4475.53 after 18 yr.

Got It? **2.** Suppose the account in Problem 2 pays interest compounded monthly. What will the account balance be after 18 yr?

The function $y = a \cdot b^x$ can model *exponential decay* as well as exponential growth. In both cases, b is determined by the percent rate of change. The value of b tells if the equation models exponential growth or decay.

take note

Key Concept Exponential Decay

Definitions

Exponential decay can be modeled by the function $y = a \cdot b^x$, where $a > 0$ and $0 < b < 1$. The base b is the **decay factor,** which equals 1 minus the percent rate of change expressed as a decimal.

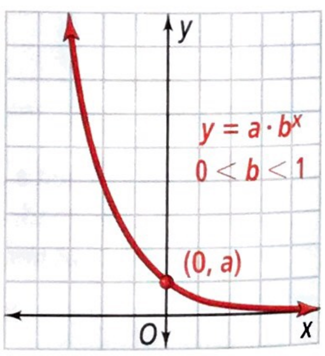
Algebra

initial amount (when $x = 0$) ↓

$y = a \cdot b^x$ ← exponent

↑ The base is the decay factor.

Graph



Problem 3 Modeling Exponential Decay STEM

Physics **The kilopascal is a unit of measure for atmospheric pressure. The atmospheric pressure at sea level is about 101 kilopascals. For every 1000-m increase in altitude, the pressure decreases about 11.5%. What is the approximate pressure at an altitude of 3000 m?**

> **Think**
> **Will the pressure ever be negative?**
> No. The range of an exponential decay function is all positive real numbers. The graph of an exponential decay function approaches but does not cross the x-axis.

Relate $y = a \cdot b^x$ Use an exponential function.

Define Let x = the altitude (in thousands of meters).
Let y = the atmospheric pressure (in kilopascals).
Let a = the initial pressure (in kilopascals), 101.
Let b = the decay factor, which is $1 - 0.115 = 0.885$.

Write $y = 101 \cdot 0.885^x$

Use the equation to estimate the pressure at an altitude of 3000 m.

$y = 101 \cdot 0.885^x$

$= 101 \cdot 0.885^3$ Substitute 3 for x.

≈ 70 Round to the nearest kilopascal.

The pressure at an altitude of 3000 m is about 70 kilopascals.

Got It? 3. a. What is the atmospheric pressure at an altitude of 5000 m?

b. Reasoning Why do you subtract the percent decrease from 1 to find the decay factor?

Lesson Check

Do you know HOW?

1. What is the growth factor in the equation $y = 34 \cdot 4^x$?
2. What is the initial amount in the function $y = 15 \cdot 3^x$?
3. What is the decay factor in the function $y = 17 \cdot 0.2^x$?
4. A population of fish in a lake decreases 6% annually. What is the decay factor?
5. Suppose your friend's parents invest \$20,000 in an account paying 5% interest compounded annually. What will the balance be after 10 yr?

Do you UNDERSTAND? MATHEMATICAL PRACTICES

6. **Vocabulary** How can you tell if an exponential function models growth or decay?
7. **Reasoning** How can you simplify the compound interest formula when the interest is compounded annually? Explain.
8. **Error Analysis** A student deposits \$500 into an account that earns 3.5% interest compounded quarterly. Describe and correct the student's error in calculating the account balance after 2 yr.

$$A = 500\left(1 + \frac{3.5}{4}\right)^{4 \cdot 2}$$
$$= 500(1.875)^8$$
$$\approx 76{,}380.09$$

Practice and Problem-Solving Exercises

MATHEMATICAL PRACTICES

A Practice

Identify the initial amount a and the growth factor b in each exponential function.

See Problem 1.

9. $g(x) = 14 \cdot 2^x$

10. $y = 150 \cdot 1.0894^x$

11. $y = 25{,}600 \cdot 1.01^x$

12. $f(t) = 1.4^t$

13. **College Enrollment** The number of students enrolled at a college is 15,000 and grows 4% each year.
    a. The initial amount a is ■.
    b. The percent rate of change is 4%, so the growth factor b is 1 + ■ = ■.
    c. To find the number of students enrolled after one year, you calculate 15,000 • ■.
    d. Complete the equation $y =$ ■•■ to find the number of students enrolled after x years.
    e. Use your equation to predict the number of students enrolled after 25 yr.

14. **Population** A population of 100 frogs increases at an annual rate of 22%. How many frogs will there be in 5 years? Write an expression to represent the equivalent monthly population increase rate.

Find the balance in each account after the given period.

See Problem 2.

15. \$4000 principal earning 6% compounded annually, after 5 yr

16. \$12,000 principal earning 4.8% compounded annually, after 7 yr

17. \$500 principal earning 4% compounded quarterly, after 6 yr

18. \$20,000 deposit earning 3.5% compounded monthly, after 10 yr

19. \$5000 deposit earning 1.5% compounded quarterly, after 3 yr

20. \$13,500 deposit earning 3.3% compounded monthly, after 1 yr

21. \$775 deposit earning 4.25% compounded annually, after 12 yr

22. \$3500 deposit earning 6.75% compounded monthly, after 6 months

Identify the initial amount a and the decay factor b in each exponential function.

See Problem 3.

23. $y = 5 \cdot 0.5^x$

24. $f(x) = 10 \cdot 0.1^x$

25. $g(x) = 100\left(\frac{2}{3}\right)^x$

26. $y = 0.1 \cdot 0.9^x$

27. **Population** The population of a city is 45,000 and decreases 2% each year. If the trend continues, what will the population be after 15 yr?

B Apply

State whether the equation represents *exponential growth, exponential decay,* or *neither.*

28. $y = 0.93 \cdot 2^x$

29. $y = 2 \cdot 0.68^x$

30. $y = 68 \cdot x^2$

31. $y = 68 \cdot 0.2^x$

32. Sports In a single-elimination tournament starting with 128 teams, half of the remaining teams are eliminated in each round.

a. Make a table, a scatter plot, and a function rule to represent the situation.

b. Is it possible for 24 teams to remain after a round? Which representation in part (a) made it the easiest to answer the question?

c. What is the domain of the function? What does the domain represent?

d. How many teams will be left after 5 rounds?

33. Car Value A family buys a car for $20,000. The value of the car decreases about 20% each year. After 6 yr, the family decides to sell the car. Should they sell it for $4000? Explain.

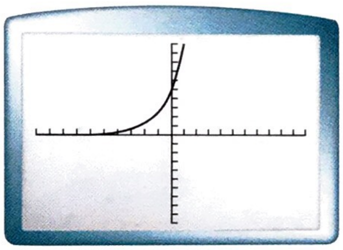
34. Think About a Plan You invest $100 and expect your money to grow 8% each year. About how many years will it take for your investment to double?

- What function models the growth of your investment?
- How can you use a table to find the approximate amount of time it takes for your investment to double?
- How can you use a graph to find the approximate amount of time it takes for your investment to double?

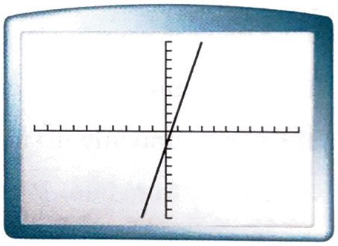
35. Reasoning Give an example of an exponential function in the form $y = a \cdot b^x$ that is neither an exponential growth function nor an exponential decay function. Explain your reasoning.

State whether each graph shows an *exponential growth function,* an *exponential decay function,* or *neither.*

36.



37.



38. Use a table and a scatter plot to answer each question.

a. You play a game of musical chairs in which 32 players start and you remove 2 chairs in each round. How many rounds will you play before two players are left?

b. In another game of musical chairs, you take away half of the chairs each time. If the game begins with 32 players, how many rounds will it take to get down to two players?

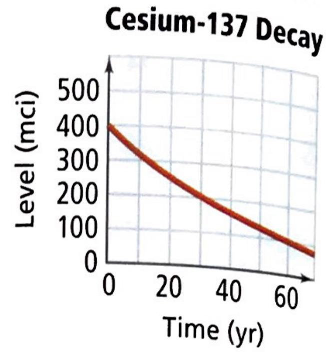
c. Will a game where you remove half of the chairs always end more quickly than one in which you take the same number of chairs each time? Give an example.

39. Business Suppose you start a lawn-mowing business and make a profit of $400 in the first year. Each year, your profit increases 5%.

a. Write a function that models your annual profit.

b. If you continue your business for 10 yr, what will your *total* profit be?

STEM **40. Medicine** Cesium-137 is a radioisotope used in radiology where levels are measured in millicuries (mci). Use the graph at the right. What is a reasonable estimate of the half-life of cesium-137?



Challenge

41. Credit Suppose you use a credit card to buy a new suit for \$250. If you do not pay the entire balance after one month, you are charged 1.8% monthly interest on your account balance. Suppose you can make a \$30 payment each month.

a. What is your balance after your first monthly payment?

b. How much interest are you charged on the remaining balance after your first payment?

c. What is your balance just before you make your second payment?

d. What is your balance after your second payment?

e. How many months will it take for you to pay off the entire bill?

f. How much interest will you have paid in all?

42. Open-Ended Write two exponential growth functions $f(x)$ and $g(x)$ such that $f(x) < g(x)$ for $x < 3$ and $f(x) > g(x)$ for $x > 3$.

PERFORMANCE TASK

Apply What You've Learned

MATHEMATICAL PRACTICES
MP 4

Use the compound interest formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ and the information on page 417 about the two CDs that Emilio is considering. Select all of the following that are true. Explain your reasoning.

A. The compound interest formula is an exponential growth function.

B. The compound interest formula is an exponential decay function.

C. For the CD from Bank West, $n = 12$.

D. For the CD from First Bank, $nt = 60$.

E. The growth/decay factor for the CD from Bank West is $\left(1 + \frac{0.038}{4}\right)^{24}$.

F. The growth/decay factor for the CD from Bank West is $\left(1 + \frac{0.038}{4}\right)^{4}$.

G. The growth/decay factor for the CD from First Bank is $0.0031\overline{6}$.