9-6 The Quadratic Formula and the Discriminant

Common Core State Standards

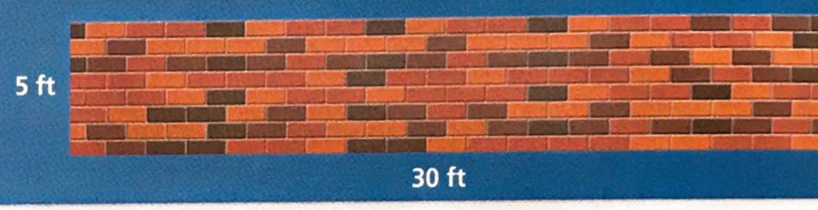
A-REI.B.4a Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ . . . Derive the quadratic formula from this form. **Also A-REI.B.4b**

MP 1, MP 2, MP 3, MP 4, MP 7, MP 8

Objectives To solve quadratic equations using the quadratic formula
To find the number of solutions of a quadratic equation

SOLVE IT! **Getting Ready!**

Your friend's aunt has a brick walkway in her backyard. Her plan is to decrease the length by the same amount she increases the width to make a rectangular patio. She wants the patio to have an area of 310 ft^2. Can she build a patio to meet her plan? Explain your reasoning.



MATHEMATICAL PRACTICES Recall that quadratic equations can have two, one, or no real-number solutions. A quadratic equation can never have more than two solutions.

Lesson Vocabulary
- quadratic formula
- discriminant

Essential Understanding You can find the solution(s) of *any* quadratic equation using the **quadratic formula**.

take note

Key Concept Quadratic Formula

Algebra

If $ax^2 + bx + c = 0$, and $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

Suppose $2x^2 + 3x - 5 = 0$. Then $a = 2$, $b = 3$, and $c = -5$. Therefore

$$x = \frac{-(3) \pm \sqrt{(3)^2 - 4(2)(-5)}}{2(2)}$$

Here's Why It Works If you complete the square for the general equation $ax^2 + bx + c = 0$, you can derive the quadratic formula.

Step 1 Write $ax^2 + bx + c = 0$ so the coefficient of x^2 is 1.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{Divide each side by } a.$$

Step 2 Complete the square.

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Subtract } \tfrac{c}{a} \text{ from each side.}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{Add } \left(\tfrac{b}{2a}\right)^2 \text{ to each side.}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \text{Write the left side as a square.}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \qquad \text{Multiply } -\tfrac{c}{a} \text{ by } \tfrac{4a}{4a} \text{ to get like denominators.}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{Simplify the right side.}$$

Step 3 Solve the equation for x.

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{Take square roots of each side.}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{Simplify the right side.}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{Subtract } \tfrac{b}{2a} \text{ from each side.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Simplify.}$$

This step uses the property $\sqrt{\frac{m}{n}} = \frac{\sqrt{m}}{\sqrt{n}}$, which you will study in Lesson 10-2.

Be sure to write a quadratic equation in standard form before using the quadratic formula.

Problem 1 Using the Quadratic Formula

Think

Why do you need to write the equation in standard form?
You can only use the quadratic formula with equations in the form $ax^2 + bx + c = 0$.

What are the solutions of $x^2 - 8 = 2x$? Use the quadratic formula.

$$x^2 - 2x - 8 = 0 \qquad \text{Write the equation in standard form.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Use the quadratic formula.}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-8)}}{2(1)} \qquad \text{Substitute 1 for } a, -2 \text{ for } b, \text{ and } -8 \text{ for } c.$$

$$x = \frac{2 \pm \sqrt{36}}{2} \qquad \text{Simplify.}$$

$$x = \frac{2 + 6}{2} \quad \text{or} \quad x = \frac{2 - 6}{2} \qquad \text{Write as two equations.}$$

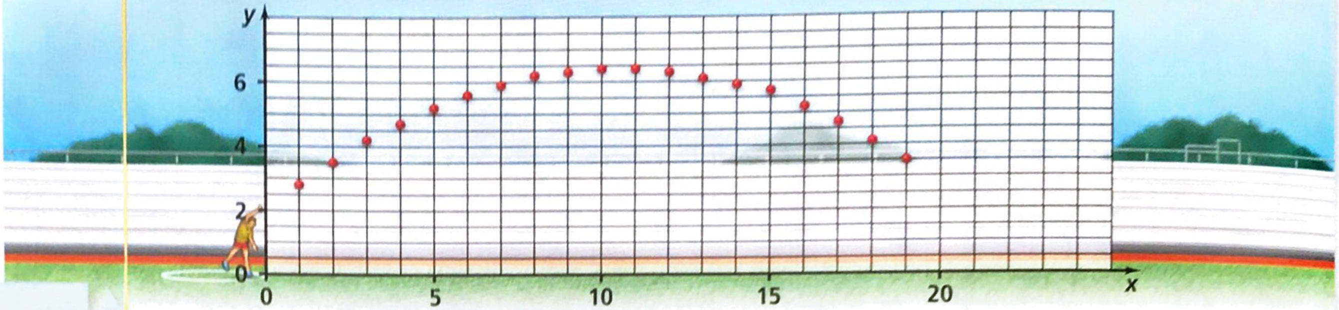
$$x = 4 \quad \text{or} \quad x = -2 \qquad \text{Simplify.}$$

Got It? **1.** What are the solutions of $x^2 - 4x = 21$? Use the quadratic formula.

When the radicand in the quadratic formula is not a perfect square, you can use a calculator to approximate the solutions of an equation.

Problem 2 Finding Approximate Solutions

Sports **In the shot put, an athlete throws a heavy metal ball through the air. The arc of the ball can be modeled by the equation $y = -0.04x^2 + 0.84x + 2$, where x is the horizontal distance, in meters, from the athlete and y is the height, in meters, of the ball. How far from the athlete will the ball land?**



Think

Why do you substitute 0 for *y*?
When the ball hits the ground, its height will be 0.

$0 = -0.04x^2 + 0.84x + 2$ — Substitute 0 for *y* in the given equation.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ — Use the quadratic formula.

$x = \dfrac{-0.84 \pm \sqrt{0.84^2 - 4(-0.04)(2)}}{2(-0.04)}$ — Substitute -0.04 for *a*, 0.84 for *b*, and 2 for *c*.

$x = \dfrac{-0.84 \pm \sqrt{1.0256}}{-0.08}$ — Simplify.

$x = \dfrac{-0.84 + \sqrt{1.0256}}{-0.08}$ or $x = \dfrac{-0.84 - \sqrt{1.0256}}{-0.08}$ — Write as two equations.

$x \approx -2.16$ or $x \approx 23.16$ — Simplify.

Only the positive answer makes sense in this situation. The ball will land about 23.16 m from the athlete.

Got It? **2.** A batter strikes a baseball. The equation $y = -0.005x^2 + 0.7x + 3.5$ models its path, where x is the horizontal distance, in feet, the ball travels and y is the height, in feet, of the ball. How far from the batter will the ball land? Round to the nearest tenth of a foot.

There are many methods for solving a quadratic equation.

Method	**When to Use**
Graphing	Use if you have a graphing calculator handy.
Square roots	Use if the equation has no x-term.
Factoring	Use if you can factor the equation easily.
Completing the square	Use if the coefficient of x^2 is 1, but you cannot easily factor the equation.
Quadratic formula	Use if the equation cannot be factored easily or at all.

Problem 3 Choosing an Appropriate Method

Think

Can you use the quadratic formula to solve part (A)?

Yes. You can use the quadratic formula with $a = 3$, $b = 0$, and $c = -9$. However, it is faster to use square roots.

Which method(s) would you choose to solve each equation? Explain your reasoning.

A $3x^2 - 9 = 0$ — Square roots; there is no x-term

B $x^2 - x - 30 = 0$ — Factoring; the equation is easily factorable

C $6x^2 + 13x - 17 = 0$ — Quadratic formula, graphing; the equation cannot be factored

D $x^2 - 5x + 3 = 0$ — Quadratic formula, completing the square, or graphing; the coefficient of the x^2-term is 1, but the equation cannot be factored

E $-16x^2 - 50x + 21 = 0$ — Quadratic formula, graphing; the equation cannot be factored easily since the numbers are large

Got It? 3. Which method(s) would you choose to solve each equation? Justify your reasoning.

a. $x^2 - 8x + 12 = 0$ **b.** $169x^2 = 36$ **c.** $5x^2 + 13x - 1 = 0$

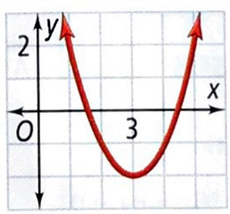
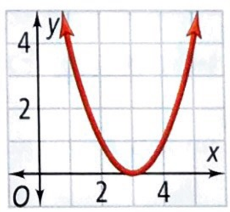
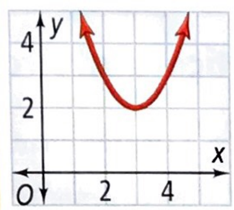
Quadratic equations can have two, one, or no real-number solutions. Before you solve a quadratic equation, you can determine how many real-number solutions it has by using the discriminant. The **discriminant** is the expression under the radical sign in the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ ← the discriminant ($b^2 - 4ac$)

The discriminant of a quadratic equation can be positive, zero, or negative.

take note

Key Concept Using the Discriminant

Discriminant	$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
Example	$x^2 - 6x + 7 = 0$ The discriminant is $(-6)^2 - 4(1)(7) = 8$, which is positive. 	$x^2 - 6x + 9 = 0$ The discriminant is $(-6)^2 - 4(1)(9) = 0$. 	$x^2 - 6x + 11 = 0$ The discriminant is $(-6)^2 - 4(1)(11) = -8$, which is negative. 
Number of Solutions	There are two real-number solutions.	There is one real-number solution.	There are no real-number solutions.

Problem 4 Using the Discriminant

How many real-number solutions does $2x^2 - 3x = -5$ have?

Plan

Can you solve this problem another way?
Yes. You could actually solve the equation to find any solutions. However, you only need to know the number of solutions, so use the discriminant.

Think	Write
Write the equation in standard form.	$2x^2 - 3x + 5 = 0$
Evaluate the discriminant by substituting 2 for *a*, −3 for *b*, and 5 for *c*.	$b^2 - 4ac = (-3)^2 - 4(2)(5)$ $= -31$
Draw a conclusion.	Because the discriminant is negative, the equation has no real-number solutions.

Got It? 4. a. How many real-number solutions does $6x^2 - 5x = 7$ have?
b. Reasoning If a is positive and c is negative, how many real-number solutions will the equation $ax^2 + bx + c = 0$ have? Explain.

Lesson Check

Do you know HOW?

Use the quadratic formula to solve each equation. If necessary, round answers to the nearest hundredth.

1. $-3x^2 - 11x + 4 = 0$
2. $7x^2 - 2x = 8$
3. How many real-number solutions does the equation $-2x^2 + 8x - 5 = 0$ have?

Do you UNDERSTAND? MATHEMATICAL PRACTICES

4. **Vocabulary** Explain how the discriminant of the equation $ax^2 + bx + c = 0$ is related to the number of x-intercepts of the graph of $y = ax^2 + bx + c$.
5. **Reasoning** What method would you use to solve the equation $x^2 + 9x + c = 0$ if $c = 14$? If $c = 7$? Explain.
6. **Writing** Explain how completing the square is used to derive the quadratic formula.

Practice and Problem-Solving Exercises MATHEMATICAL PRACTICES

A Practice

Use the quadratic formula to solve each equation. See Problem 1.

7. $2x^2 + 5x + 3 = 0$
8. $5x^2 + 16x - 84 = 0$
9. $4x^2 + 7x - 15 = 0$
10. $3x^2 - 41x = -110$
11. $18x^2 - 45x - 50 = 0$
12. $3x^2 + 44x = -96$
13. $3x^2 + 19x = 154$
14. $2x^2 - x - 120 = 0$
15. $5x^2 - 47x = 156$

Use the quadratic formula to solve each equation. Round your answer to the nearest hundredth.

See Problem 2.

16. $x^2 + 8x + 11 = 0$
17. $5x^2 + 12x - 2 = 0$
18. $2x^2 - 16x = -25$
19. $8x^2 - 7x - 5 = 0$
20. $6x^2 + 9x = 32$
21. $3x^2 + 5x = 4$

22. **Football** A football player punts a ball. The path of the ball can be modeled by the equation $y = -0.004x^2 + x + 2.5$, where x is the horizontal distance, in feet, the ball travels and y is the height, in feet, of the ball. How far from the football player will the ball land? Round to the nearest tenth of a foot.

Which method(s) would you choose to solve each equation? Justify your reasoning.

See Problem 3.

23. $x^2 + 4x - 15 = 0$
24. $9x^2 - 49 = 0$
25. $4x^2 - 41x = 73$
26. $3x^2 - 7x + 3 = 0$
27. $x^2 + 4x - 60 = 0$
28. $-4x^2 + 8x + 1 = 0$

Find the number of real-number solutions of each equation.

See Problem 4.

29. $x^2 - 2x + 3 = 0$
30. $x^2 + 7x - 5 = 0$
31. $x^2 + 3x + 11 = 0$
32. $x^2 - 15 = 0$
33. $x^2 + 2x = 0$
34. $9x^2 + 12x + 4 = 0$

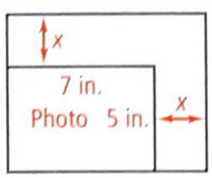
B Apply

Use any method to solve each equation. If necessary, round your answer to the nearest hundredth.

35. $3w^2 = 48$
36. $3x^2 + 2x - 4 = 0$
37. $6g^2 - 18 = 0$
38. $3p^2 + 4p = 10$
39. $k^2 - 4k = -4$
40. $13r^2 - 117 = 0$

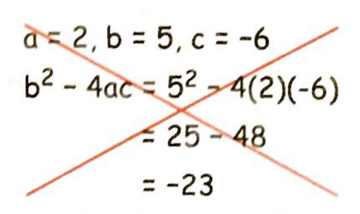
41. **Think About a Plan** You operate a dog-walking service. You have 50 customers per week when you charge \$14 per walk. For each \$1 decrease in your fee for walking a dog, you get 5 more customers per week. Can you ever earn \$750 in a week? Explain.
    - What quadratic equation in standard form can you use to model this situation?
    - How can the discriminant of the equation help you solve the problem?

42. **Sports** Your school wants to take out an ad in the paper congratulating the basketball team on a successful season, as shown below. The area of the photo will be half the area of the entire ad. What is the value of x?



43. **Writing** How can you use the discriminant to write a quadratic equation that has two solutions?

44. **Error Analysis** Describe and correct the error at the right that a student made in finding the discriminant of $2x^2 + 5x - 6 = 0$.



45. Find the discriminant and the solution of each equation in parts (a)–(c). If necessary, round to the nearest hundredth.

a. $x^2 - 6x + 5 = 0$ b. $x^2 + x - 20 = 0$ c. $2x^2 - 7x - 3 = 0$

d. **Reasoning** When the discriminant is a perfect square, are the solutions rational or irrational? Explain.

Challenge

46. **Reasoning** The solutions of any quadratic equation $ax^2 + bx + c = 0$ are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

a. Find a formula for the sum of the solutions.

b. One solution of $2x^2 + 3x - 104 = 0$ is -8. Use the formula you found in part (a) to find the second solution.

Reasoning For each condition given, tell whether $ax^2 + bx + c = 0$ will *always, sometimes,* or *never* have two solutions.

47. $b^2 < 4ac$ 48. $b^2 = 0$ 49. $ac < 0$

Standardized Test Prep

SAT/ACT

50. What are the approximate solutions of the equation $x^2 - 7x + 3 = 0$?

(A) $-6.54, 0.46$ (B) $-6.54, -0.46$ (C) $-0.46, 6.54$ (D) $0.46, 6.54$

51. Which of the following relations is a function?

(F) $\{(1, 2), (3, 5), (1, 4), (2, 3)\}$

(G) $\{(-5, 6), (0, 9), (-1, 2), (0, 6)\}$

(H) $\{(8, 2), (6, 3), (6, 11), (-8, 2)\}$

(I) $\{(-1, 3), (7, 3), (-7, 2), (4, 5)\}$

52. What equation do you get when you solve $3a - b = 2c$ for b?

(A) $b = -3a + 2c$ (B) $b = 3a - 2c$ (C) $b = 3a + 2c$ (D) $b = -3a - 2c$

53. What are the approximate solutions of the equation $\frac{1}{3}x^2 - \frac{5}{4}x + 1 = 0$? Use a graphing calculator.

(F) 1.07, 2.77 (G) 1.16, 2.59 (H) 0.87, 10.38 (I) 0.19, 16.01

Short Response

54. Suppose the line through points $(n, 6)$ and $(1, 2)$ is parallel to the graph of $2x + y = 3$. Find the value of n. Show your work.

Mixed Review

Solve each equation by completing the square.

See Lesson 9-5.

55. $s^2 - 10s + 13 = 0$ 56. $m^2 + 3m = -2$ 57. $3w^2 + 18w - 1 = 0$

Get Ready! **To prepare for Lesson 9-7, do Exercises 58–61.**

Graph each function.

See Lesson 7-6.

58. $y = 2^x$ 59. $y = 3^x$ 60. $y = \left(\frac{1}{3}\right)^x$ 61. $y = \left(\frac{1}{2}\right)^x$